Semigroups with a strong product

Shanghai, April 2012

<u>Motivation</u>: ssa algebras

<u>Defn</u> (Toms-Winter, 07)

A unital, separable, C^*-algebra is ssa if $D \ncong \mathbb{C}$ and

$$\exists\, \varphi : D \to D \otimes D \quad \text{st} \quad \varphi \approx_{a.u.} \mathrm{id}_D \otimes 1_D : \begin{array}{l} D \to D\otimes D \\ d \mapsto d \otimes 1 \end{array}$$

($\varphi \approx_{a.u.} \psi$ if $\exists\, (u_n)$ unitaries such that $u_n \varphi(a) u_n^* \to \psi(a)$ $\forall a \in A$)

We could as well have defined ssa by requiring $\varphi \approx_{a.u.} 1_D \otimes \mathrm{id}_D$, in which case we obtain an equivalent defn, because if we put $\sigma : \begin{array}{l} D\otimes D \to D \otimes D \\ a\otimes b \mapsto b \otimes a \end{array}$ (the flip) ~~we get $\sigma\circ\varphi$~~ and $\varphi \approx_{a.u.} \mathrm{id}_D \otimes 1_D$, then

$$\sigma \circ \varphi \approx_{a.u.} \sigma(\mathrm{id}_D \otimes 1_D) = 1_D \otimes \mathrm{id}_D.$$

<u>Thm</u> (Toms-Winter, 07) If D is ssa, then is nuclear and simple. Also, either D is purely infinite or stably finite with a unique trace.

<u>Examples</u>: $\mathcal{Z}$, the Jiang-Su algebra, any UHF algebra of infinite type, $\mathcal{O}_2$, $\mathcal{O}_\infty$, $\mathcal{O}_2 \otimes \mathcal{O}_\infty$, UHF of infinite type $\otimes\, \mathcal{O}_\infty$.

All these algebras happen to be $\mathcal{Z}$-stable.

Defn: A is $\mathcal{Z}$-stable provided $A \cong A \otimes \mathcal{Z}$. More generally for a D ssa, A is called D-stable if $A \cong A \otimes D$.
Notice that, given A, $A \otimes D$ is always D-stable if D is ssa
This prompted the Question: D ssa $\Rightarrow$ D $\mathcal{Z}$-stable?

Thm 1 (Dadarlat-Rørdam, 09): Yes, if D contains a non-trivial projection. In this case, D has moreover RRO.

Thm 2 (Winter, 11): Yes, in full generality.

This is related to

Question: Are the previous examples the only ones?

Thm (Dadarlat-Toms, 10): The only ASH algebra which is ssa and projectionless is $\mathcal{Z}$.

[Maybe mention: Does there exist a simple, sep, nuclear C^*-algebra which is not ASH?]

We want first to look at some of these results from the point of view of the Cuntz semigroup.

Defn: For a C^*-algebra A, its Cuntz semigroup is

$Cu(A) = \frac{(A\otimes K)_+}{\sim}$, where $a \sim b$ if $a \precsim b$ and $b \precsim a$;

and $a \precsim b$ if $a = \lim x_n b x_n^*$, for a sequence (x_n) in
$A \otimes K$

$Cu(A)$ is a semigroup with operation

$$[a]+[b]=\left[\varphi\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\right]$$

where $\varphi : M_2(A\otimes K) \to A\otimes K$ is an (inner) isomorphism.
This operation is abelian, and compatible with an order, defined by $[a]\leq[b]$ if $a\precsim b$.
The structure of Cu is richer, as was shown by Coward-~~Elliott~~-Ivanescu (08)

<u>Defn</u>: The category Cu is a category whose objects are ordered semigroups such that, for each $S\in Cu$, ~~every~~

- Every increasing sequence in S has a supremum.
- Every elements in S is a supremum $s=\sup x_n$

with $x_n \ll x_{n+1}$ (such a sequence is called rapidly increasing).
[Here, $x\ll y$ means: if $y\leq \sup z_n$, then $\exists m: x\leq z_m$.]
Maps in Cu are semigroup maps that preserve all the structure.
It seems pertinent to add two more requirements to the objects in Cu:

(AAO) - almost algebraic order: $x\leq y$ & $x'\ll x \Rightarrow \exists\, t$ such that $x'+t\leq y\leq x+t$.

(AR) - almost Riesz Decomposition: $x'\ll x\leq y+z \Rightarrow \exists\, y'\leq y, x$ and $z'\leq z, x$ with $x'\leq y'+z'$.

<u>Thm</u> (Coward-Elliott-Ivanescu, 08; Rørdam-Winter, 10; Robert, 12):
For any C^*-algebra A, $Cu(A)\in Cu$.

Note here that the typical example of $\ll$ is
$[(a-\epsilon)_+] \ll [a]$ $\forall a$ +ve and $\epsilon > 0$.

It is known that, whenever A is p.i. simple, then $Cu(A)$ is degenerate, namely $\{0, \infty\}$, so we shall restrict to stably finite examples.

~~Proposition: If A, B are st.~~

Defn: Given S, T, R in Cu, a $\ll$-bimorphism is a biadditive map $\varphi: S \times T \to R$ that preserves suprema of increasing sequences in each variable, and such that $\varphi(s', t') \ll \varphi(s, t)$ whenever $s' \ll s$ and $t' \ll t$.

Proposition: If A, B are stable nuclear C^*-algebras, then the natural map $A \times B \to A \otimes B$, $(a, b) \mapsto a \otimes b$ induces a $\ll$-bimorphism
$Cu(A) \times Cu(B) \xrightarrow{\alpha} Cu(A \otimes B)$, $([a], [b]) \mapsto [a \otimes b]$

Sketch: If w_1, w_2 are isometries in $M(A)$ w/ orthogonal ranges, then for $a, a' \in A$, $b \in B$:
$$[(w_1 a w_1^* + w_2 a' w_2^*) \otimes b] = [w_1 a w_1^* \otimes b] + [w_2 a' w_2^* \otimes b] = [a \otimes b] + [a' \otimes b]$$
$\Rightarrow$ biadditive

By comparing norms of $a \otimes b$ and $(a-\epsilon)_+ \otimes (b-\epsilon)_+$, easy to see that $[a \otimes b] = \sup_{\epsilon > 0} [(a-\epsilon)_+ \otimes (b-\epsilon)_+]$

So, if $\sup[a_n] = [a]$, $\forall n$ have $[a_n \otimes b] \leq [a \otimes b]$

Given $\varepsilon > 0$, $\exists n$: $[(a-\varepsilon)_+] \leq [a_n]$, so

$$[(a-\varepsilon)_+ \otimes (b-\varepsilon)_+] \leq [a_n \otimes b] \leq \sup [a_n \otimes b]$$

$$\Rightarrow [a \otimes b] \leq \sup [a_n \otimes b].$$

Finally: $[a'] \ll [a]$, $[b'] \ll [b]$ $\Rightarrow$ $[a'] \leq [(a-\varepsilon)_+]$, $[b'] \leq [(b-\varepsilon)_+]$

$\Rightarrow [a' \otimes b'] \leq [(a-\varepsilon)_+ \otimes (b-\varepsilon)_+]$.

$$(a-\varepsilon)_+ \otimes (b-\varepsilon)_+ \in A \otimes B \subseteq M(A) \otimes M(B)$$

In (multipliers): $(a-\varepsilon)_+ \otimes (b-\varepsilon)_+ = ((a-\varepsilon)_+ \otimes 1)(1 \otimes (b-\varepsilon)_+)$

and $[(a-\varepsilon)_+ \otimes 1] \ll [a \otimes 1]$ in $Cu(M(A) \otimes M(B))$

$[1 \otimes (b-\varepsilon)_+] \ll [1 \otimes b]$ in $Cu(M(A) \otimes M(B))$

$\Rightarrow$ $[(a-\varepsilon)_+ \otimes 1] \leq [(a \otimes 1 - \varepsilon')_+]$ Since all elts commute, get:

$[1 \otimes (b-\varepsilon)_+] \leq [(1 \otimes b - \varepsilon')_+]$

$$[((a-\varepsilon)_+ \otimes 1)(1 \otimes (b-\varepsilon)_+)] \leq [(a \otimes 1 - \varepsilon')_+ (1 \otimes b - \varepsilon')_+] \leq [(a \otimes b - (\varepsilon')^2)_+]$$

$\ll [a \otimes b]$. ∎

Next, if D is ssa and write $D \xrightarrow{\varphi} D \otimes D$ so we may

define $Cu(D) \times Cu(D) \xrightarrow{\alpha} Cu(D \otimes D) \xrightarrow{Cu(\varphi^{-1})} Cu(D)$

$([a], [b]) \longmapsto [a \otimes b] \longmapsto [\varphi^{-1}(a \otimes b)]$

which is a $\ll$-bimorphism composed w/ a map in Cu,

whence defines a product, distributive, associative, compatible

with suprema and ~~and that~~ with $\ll$.

Note the product is commutative:

D has approximately inner flip if $\sigma_D \approx_{a.u.} \mathrm{id}_{D\otimes D}$.

Thm (Toms–Winter, 07) If D is ssa $\Rightarrow$ $\sigma_D \approx_{a.u.} \mathrm{id}_{D\otimes D}$.

This implies that $a\otimes b \sim b\otimes a$, whence $\varphi^{-1}(a\otimes b) \sim \varphi^{-1}(b\otimes a)$.

so $Cu(D)$ is commutative.

Also notice that

$$\|1\otimes b - u_n \varphi(b) u_n^*\| \to 0 \Rightarrow \|\varphi^{-1}(u_n)(b)\varphi^{-1}(u_n^*) - \varphi^{-1}(1\otimes b)\| \to 0$$

$\Rightarrow \varphi^{-1}(1\otimes b) \sim b$. This implies $Cu(D)$ unital

Def: A semigroup $S \in Cu$ has a strong product if

$\exists$ $\ll$-bimorphism $\begin{array}{c} S\times S \to S \\ (a,b) \mapsto ab \end{array}$ associative and commutative

and $\exists\, 1 \ll 1$ which is a unit for the product.

We moreover assume S satisfies a universal property for functionals, namely, if $S\times S \xrightarrow{\tau} [0,\infty]$ is a map st $\tau(a,b) = \tau(a,1)\tau(1,b)$, then $\exists!\, \tilde\tau: S \to [0,\infty]$ such that

$$\begin{tikzcd}
S\times S \arrow[r, "\mathrm{prod.}"] \arrow[d, "\tau"'] & S \arrow[dl, "\tilde\tau"] \\
{[0,\infty]} &
\end{tikzcd} \quad \text{commutes}$$

This property can be verified for $S = Cu(D)$, D ssa, and it implies S simple with a unique functional.

L. Robert (2011) defined, for $S \in Cu$, the concept of real multiplication, as $(0,\infty] \times S \to S$, biadditive, order preserving, suprema preserving, and $1.s = s$ $\forall s$.

Showed that if $S \in Cu$, $\exists$ S_R with real multiplication and maximal with this property (so $(S_R)_R \cong S_R$).

Jacelon studied in 2010 (also Robert in 2012) a C^*-algebra R non-unital, stably projectionless, such that $R \otimes R \cong R$, $R \otimes Q \cong R$ (Q is the UHF algebra with $K_0(Q) \cong \mathbb{Q}$).

Thm (Robert): For any C^*-algebra $Cu(A \otimes R) \cong Cu(A)_R$.

Aim: Try to do something along these lines for $\mathcal{Z}$ instead of R. In more general circumstances, for $\mathcal{D}$ ssa.

We are not quite there yet, but we have some results we believe have some interest.

First, if we consider $A \cong A \otimes \mathcal{Z}$, we have a Cu-bimorphism

$$Cu(\mathcal{Z}) \times Cu(A) \to Cu(\mathcal{Z} \otimes A) \cong Cu(A)$$

We say that $Cu(A)$ has $Cu(\mathcal{Z})$-multiplication.

~~In general, S has Z~~

$Cu(\mathcal{Z}) \cong \mathbb{N}_0 \sqcup (0,\infty] = Z$.

In general, $S \in Cu$ has Z-multiplication if $\exists$ a Cu-bimorphism $Z \times S \to S$ with $1_Z \cdot s = s$

~~Z-multiplication~~ It is known that any $\mathcal{Z}$-stable C^*-algebra A satisfies that $Cu(A)$ is <u>almost unperforated</u> and <u>almost divisible</u>:

almost unperforated: $(k+1)x \leq ky$ some $k \Rightarrow x \leq y$.

almost divisible: $\forall x, \forall k, \exists y$: $ky \leq x \leq (k+1)y$.

<u>Thm</u>: $S \in Cu$, then S has Z-multiplication iff S is almost unperforated and almost divisible

<u>Pf</u>: Show $\Rightarrow$) (easy part).

$$(n+1)x \leq ny \Rightarrow \underbrace{\tfrac{1}{n}x + \overset{(n+1)}{\cdots} + \tfrac{1}{n}x}_{(1'+1/n)x} \leq 1/n\, y + \overset{(n}{\cdots} + 1/n\, y = 1'y.$$

$$\Rightarrow x \leq (1'+1/n)x \leq 1'y \leq y.$$

Given x, n:

$$n.\left(\tfrac{1}{n}x\right) = 1'x \leq x \leq (1'+1/n)x = (n+1)(1/n\, x), \text{ so take } y = \tfrac{1}{n}x \quad \blacksquare$$

Just as was done for simple $\mathcal{Z}$-stable C^*-algebras, we can give a representation thm for simple semigroups in Cu with Z-multiplication (and that have a distinguished compact elt, not zero):

How to do this?

Such an S has real multiplication, by restriction of scalars. In fact, $S = S_c \sqcup 1'S$, and $1'S$ has real multiplication. So this allows to define:

$$\begin{aligned} S &\to S_c \sqcup LAff(F(S)_e)^{++} \\ s &\mapsto s \text{ if } s \in S_c \\ s &\longmapsto F(S)_e \to \mathbb{R}^{++} \cup \{\infty\} \text{ if } s \notin S_c. \\ &\qquad \lambda \longmapsto \lambda(s). \end{aligned}$$

~~In the case of one functional~~

(That this is iso follows from representing $1'S \cong L(F(1'S))$ by Robert's thm, and identifying $L(F(1'S))$ with $LAff(F(S)_e)^{++}$)

In the case of one functional, this yields

$$S \cong S_c \sqcup (0,\infty].$$

This will be the case for S with a strong product, provided they have $\mathbb{Z}$-multiplication.

Is this always the case?

Thm: $S \in Cu$ with a strong product. Then S has $\mathbb{Z}$-multiplication. $S \ne \mathbb{N}$

Sketch: ① Enough to almost divide 1.

If $n z \leq 1 \leq (n+1) z$ then $n(zs) \leq s \leq (n+1)(zs)$

and $(n+1)s \leq nt \Rightarrow s \leq (n+1) zs \leq n z t \leq t$.

(2) $a \le k \cdot 1$, $0 < b << b' < 1$ $\Rightarrow \exists\, \ell \ge 0$ st $ab^{\ell} \le 1$.

Find $t \neq 0$: $b + t \le 1 \le b' + t \to b^2 + bt \le b$, and so

$b^2 + bt + t \le b + t \le 1$. In this way:

$$b^n + t(b^{n-1} + \dots + b + 1) \le 1$$

Find m: $b \le mt$ and let $\ell = km + 1$

$$ab^{\ell} = ab^{km} \cdot b \le a\, b^{km}\, mt \le km\, b^{km} \cdot t$$
$$\le (b^{km} + b^{km-1} + \dots + b + 1)\, t \le 1.$$

(3) $\forall k$, $\exists a \neq 0$ and $b << a$ with $ka \le 1$, $ka \le (k+1)b$.

Take λ the unique functional, $0 < s << s' < 1$, s non compact

$\lambda(ks) < \lambda((k+1)s) \Rightarrow \exists\, s_1 << s$ st $\lambda(ks) < \lambda((k+1)s_1)$

Rørdam $\Rightarrow \exists n$: $kns \le (k+1)ns_1$ and even $kns << (k+1)ns_1$

$s_1 << s$

$\Rightarrow kn(ss_1) << (k+1)nss_1$

Continue in this way to find $s_1, \dots, s_\ell, \dots$ with $kn^{s} s_1 \dots s_\ell <<$ $(k+1)n s_1 \dots s_\ell$. Take $a = ns_1 \dots s_\ell$.

(4) If $\exists\, 0 \neq a \in S$, $b << a$ with $2ka \le 1$, $2ka << (2k+1)b$

then 1 is ~~almost~~ divisible.

[This is more technical, so omitted.] ∎

T-fractions, for T with a strong product, are possible when T comes from a UHF alg. of infinite type?

This is possible, yes, e.g. if $P = p^\infty$, p prime number define $S_P = \lim (S, \cdot p)$ in the category $\mathcal{C}u$.

Then one can show that S_P corresponds to the tensor product, in the category, of S with $\mathcal{C}u(UHF(P))$.

For Z, this is harder, but still.